1.8.2 (f)
1.8.4
1.8.27(b)
✓ 2.2.8
✓ 2.2.10
2.4.4
-2.4.14
✓ 2.5.21 (a) also dim range
(b) rank
dim ker

2.4.15
✓ 2.4.17 (a)
(b)
✓ 2.3.31 (a)
(b)

2.5.21

For each of the following matrices find bases

for

(i) range

(ii) corange

(iii) Kernel

(iv) coKernel

Also, what is

(v) dim range

(vi) rank

(vii) dim Kernel ?

(a) $\begin{pmatrix} 1 & -3 \\ 2 & 6 \end{pmatrix}$

(i) range $\left(\begin{array}{cc|c} 1 & -3 & a \\ 2 & 6 & b \end{array}\right) \xrightarrow{-2R1+R2} \begin{pmatrix} 1 & -3 & a \\ 0 & 0 & b-2a \end{pmatrix}$

has soln iff $b-2a = 0 \Rightarrow b = 2a$

$\Rightarrow$ range $= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid b = 2a \right\}$

$\begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is a basis

(ii) corange $A^{+} = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 2 & a \\ -3 & 6 & b \end{array}\right) \xrightarrow{3R1+R2} \begin{pmatrix} 1 & 2 & a \\ 0 & 0 & 3a+b \end{pmatrix}$

has soln iff $b = -3a$

$\Rightarrow$ corange $= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid b = -3a \right\}$

$\begin{pmatrix} a \\ -3a \end{pmatrix} = a \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$ is a basis

(iii)

$$\left(\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & 6 & 0 \end{array}\right) \xrightarrow{-2R1+R2} \left(\begin{array}{ccc} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ x_2 is free $\quad x_1 - 3x_2 = 0$

$\Rightarrow x_1 = 3x_2$

Kernel $= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_1 = 3x_2 \right\}$

$\begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\quad \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ is a basis.

(iv) cokernel

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right) \xrightarrow{3R1+R2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ x_2 free $\quad x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

cokernel $= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_1 = -2x_2 \right\}$

$\begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\quad \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ is a basis.

(v) dim range $= 1$ (always equal)

(vi) rank $= 1$

(vii) dim Kernel $= 1$

2.4.17*

Prove that the space, T, upper triangular 2×2 matrices is a 3 dimensional vector space.

(1) First prove vector space:

Let $A, B \in T$.

Then $A + B \in T$, since the sum of upper triangular matrices is upper triangular.

Let $A \in T$, $c \in \mathbb{R}$.

Then $cA \in T$ since cA will still be upper triangular.

Also 0 matrix is upper triangular.

Thus T is a subspace, and hence a vector space.

(2) Find a basis.

Let $$B = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{B_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{B_3} \right\}.$$

Let $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T$. then

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So B spans T.

Also if $c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

then $\begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow c_1 = c_2 = c_3 = 0$.

So B_1, B_2, B_3 are linearly independent.

Thus $\mathcal{B}$ is a basis for T.

3 vectors in $\mathcal{B}$ $\Rightarrow$ T is 3-dim.

2.3.31.(a)*

Prove that if $v_1, \dots, v_K$ are linearly independent, then $v_1, \dots, v_{K-1}$ are linearly independent.

Proof:

- Assume $v_1, \dots, v_K$ are linearly ind.

Suppose $c_1 v_1 + c_2 v_2 + \dots + c_{K-1} v_{K-1} = 0$

Then $c_1 v_1 + \dots + c_{K-1} v_{K-1} + 0 v_K = 0$

$\Rightarrow c_1, \dots, c_{K-1} = 0$ since $v_1, \dots, v_K$ are lin ind.

Thus $v_1, \dots, v_{K-1}$ are lin. ind.

2.2.8 Prove that the set of all solutions, x, to $Ax = b$ forms a subspace iff $b = 0$.

($0 \in$ set $\Rightarrow b$ must be zero)

Let x_1, x_2 be solns.

Then $A(x_1 + x_2) = Ax_1 + Ax_2 = b + b = 2b$

So $x_1 + x_2$ is a soln iff $2b = b \Leftrightarrow b = 0$.

Also $A(cx_1) = cAx_1 = cb$ (c arbitrary).

So cx_1 is a soln iff $cb = b \Leftrightarrow b = 0$

Thus the set of solns is a subspace iff $b = 0$.

2.2.10 Show that $T = \{ A \in M_{n \times n} \mid \operatorname{tr} A = 0 \}$ is a subspace of $M_{n \times n}$

Let $A, B \in T$.

Then $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B = 0 + 0 = 0$.

So, $A + B \in T$.

Let $A \in T$, $c \in \mathbb{R}$.

Then $\operatorname{tr}(cA) = c \operatorname{tr} A = c0 = 0$.

So $cA \in T$.

Also $\operatorname{tr}(0) = 0$. So $0 \in T$.

Thus T is a subspace of $M_{n \times n}$.

1.8.4. Let $A = \left(\begin{array}{ccc|c} a & 0 & 2 & 2 \\ a & 2 & b & b \\ b & 2 & a & a \end{array} \right)$ be the aug. matrix for a lin. system

For what values of a and b does the system have

(i). a unique soln

(ii) many solns

(iii) no solns

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{array} \right)$$

if $a = 0$ $\rightarrow \begin{pmatrix} b & 2 & 0 & 0 \\ 0 & 2 & a & b \\ 0 & 0 & b & 2 \end{pmatrix}$ if $b \neq 0$, 3 pivots $\rightarrow$ <u>unique soln</u>

if $b = 0$, $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$\rightarrow$ many solns

if $a \neq 0$ $\xrightarrow[-b/a R1 + R2]{-R1 + R2} \begin{pmatrix} a & a & b & 2 \\ 0 & 2 & a-b & b-2 \\ 0 & 2 & a - b^2/a & a - 2b/a \end{pmatrix} \xrightarrow{-R2 + R3} \begin{pmatrix} a & 0 & b & 2 \\ 0 & 2 & 0 & b-2 \\ 0 & 0 & b - b^2/a & a - b + 2 - 2b/a \end{pmatrix}$

if $b=0$

$$\left(\begin{array}{ccc|c} a & 0 & 0 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & a+2 \end{array}\right)$$

if $a=-2$ <u>many solns</u>

if $a \neq -2$ <u>no soln</u>.

$b - b^2/a = 0 \iff ab - b^2 = 0 \iff b(a-b)=0 \iff a=b$

$a - a + 2 - 2a/a = 2-1 = 1 \neq 0$

If $a=b\neq 0$, <u>no soln</u>.

otherwise if $b \neq 0$, <u>unique soln</u>.

(i) $a\neq b$, $a\neq 0$, $b \neq 0$.

$a=0$, $b\neq 0$ $\Rightarrow$ $b\neq 0$, $a\neq b$.

(ii) $a=b=0$ or $a\neq 0$, $b=0$, $a\neq -2$

$\Rightarrow$ $b=0$, $a\neq -2$.

(iii) $a=b\neq 0$ or $a\neq -2$, $b=0$.